Q. What is an equation?

An eqn. is a relationship between numbers, at least one of which is (usually) unknown

Q. What is a solution to an equation?

A soln. is a number or set of numbers you can plug in to make the LHS of the eqn. the same as the RHS.

Q. What does it mean to solve an equation?

Solving an eqn. means finding all solns.

Ex 1 Show that $t = 1$ is a solution to $t^5 - t^4 + 2t^3 - 5t^2 + 5t - 2 = t^3 - 1$.

LHS:

RHS:

Ex 2 Is $x = 2$ a solution to $x^3 - x^2 - 3x - 2 = 0$?

LHS: $2^3 - 2^2 - 3 \cdot 2 - 2 = 8 - 4 - 6 - 2 = -4$

RHS: 0

Note: The equals signs in the above examples mean...

Equality of numbers

Def: A differential equation is an equation relating some functions and their derivatives

Here are some examples:

$$\frac{dy}{dt} + 3t = y^2$$

$$\frac{d^{10}w}{dz^{10}} - 13 = \sin(w)$$

Note: The equals sign in a differential equation means...

Equality of functions

Q. What is a solution to a differential equation?

An explicit formula that can be plugged into the unknown fn. to make the LHS and the RHS be the same fn.

Ex 3 Show that $y = e^x + x^3$ is a solution to $\frac{dy}{dx} - 3x^2 = y - x^3$

$\frac{dy}{dx} = e^x + 3x^2$

LHS: $\frac{dy}{dx} - 3x^2 = (e^x + 3x^2) - 3x^2 = e^x$

RHS: $y - x^3 = (e^x + x^3) - x^3 = e^x$ → the same fn

Ex 4 Is $y = 2e^x + x^3$ a solution to $\frac{dy}{dx} - 3x^2 = y - x^3$? What about $y = e^x + 2x^3$?

$y = 2e^x + x^3$

$\frac{dy}{dx} = 2e^x + 3x^2$

LHS: $(2e^x + 3x^2) - 3x^2 = 2e^x$

RHS: $(2e^x + x^3) - x^3 = 2e^x$ → the same fn. So $y = 2e^x - x^3$ is a soln.

$y = e^x + 2x^3$

$\frac{dy}{dx} = e^x + 6x^2$

LHS: $(e^x + 6x^2) - 3x^2 = e^x + 3x^2$

RHS: $(e^x + 2x^3) - x^3 = e^x + x^3$ → Not the same fn. so $y = e^x + 2x^3$ is not a soln.

Ex 5 Solve $\frac{dy}{dx} = y$

$\rightarrow \int \frac{dy}{y} = \int dx \rightarrow \ln|y| = x + K_1$

$y = 0$

$\rightarrow |y| = e^{x+K_1} = e^x e^{K_1} = Ke^x \ (K>0)$

$\rightarrow y = Ke^x \ (K \neq 0)$

$\Rightarrow y = Ke^x$ for any $K \in \mathbb{R}$

Ex 6 Use power series to solve $\frac{dy}{dx} = y$.

Suppose $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$\downarrow$

$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

To have $y = \frac{dy}{dx}$, need $c_0 = c_1$, $c_1 = 2c_2$, $c_2 = 3c_3$,

$c_3 = 4c_4, \ldots$

$c_n = (n+1)c_{n+1}$

$\downarrow$

$c_1 = c_0$

$c_2 = \frac{c_1}{2} = \frac{c_0}{2}$

$c_3 = \frac{c_2}{3} = \frac{c_0}{2 \cdot 3}$

$c_4 = \frac{c_3}{4} = \frac{c_0}{2 \cdot 3 \cdot 4}$

$\vdots$

$c_{n+1} = \frac{c_n}{n+1} = \frac{c_0}{(n+1)!}$

$\rightarrow \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n$

$= c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= c_0 e^x$

Ex 7 Consider the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

(a) Solve the differential equation.

Suppose $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution

would rather have x^n

$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$

re-index

$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2 \cdot 1 c_2 + 3 \cdot 2 \cdot c_3 x + 4 \cdot 3 c_4 x^2 + \cdots$

$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

$\frac{d^2y}{dx^2} + y = 0 \rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$

$\rightarrow \sum_{n=0}^{\infty} \left\{ (n+2)(n+1) c_{n+2} + c_n \right\} x^n = 0$

as functions

Need: $(n+2)(n+1) c_{n+2} + c_n = 0$

$\rightarrow c_{n+2} = \frac{-c_n}{(n+2)(n+1)}$ for $n \geq 0$

(b) Suppose you know that your solution satisfies $\frac{dy}{dx}\big|_{x=0} = -7$ and $y(0) = 2$. Find an exact formula for y as a function of x.

$$\rightarrow \quad C_2 = \frac{-C_0}{2\cdot 1} \qquad C_3 = \frac{-C_1}{3\cdot 2}$$

$$C_4 = \frac{-C_2}{4\cdot 3} = \frac{C_0}{4\cdot 3\cdot 2\cdot 1} \qquad C_5 = \frac{-C_3}{5\cdot 4} = \frac{C_1}{5\cdot 4\cdot 3\cdot 2}$$

$$C_6 = \frac{-C_4}{6\cdot 5} = \frac{-C_0}{6!} \qquad C_7 = \frac{-C_5}{7\cdot 6} = \frac{-C_1}{7!}$$

$$\vdots$$

$$C_{2k} = \frac{(-1)^k C_0}{(2k)!} \qquad C_{2k+1} = \frac{(-1)^k C_1}{(2k+1)!}$$

$$\Rightarrow y = \sum_{k=0}^{\infty} \frac{(-1)^k C_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k C_1}{(2k+1)!} x^{2k+1}$$

$$= C_0 \cos(x) + C_1 \sin(x)$$

(b) $\left.\frac{dy}{dx}\right|_{x=0} = -7 \qquad y(0) = 2$

$$2 = y(0) = c_0 \cos(0) + c_1 \sin(0) = c_0 \cdot 1 + c_1 \cdot 0$$
$$= c_0$$

$$\frac{dy}{dx} = -2 \cdot \sin(x) + c_1 \cos(x)$$

$$-7 = \left.\frac{dy}{dx}\right|_{x=0} = -2 \cdot 0 + c_1 \cdot 1 = c_1$$

$$\Rightarrow y = 2\cos(x) - 7\sin(x)$$

Ex 8 Find the 5th Maclaurin polynomial for the power series solution to $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$ where $y(0) = 1$ and $\frac{dy}{dx}\big|_{x=0} = 3$.

$y = \sum_{n=0}^{\infty} c_n x^n \rightarrow 1 = y(0) = c_0$

$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1} \rightarrow x\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^n$

$\rightarrow 3 = \frac{dy}{dx}\big|_{x=0} = c_1$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ ← from before

$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$

$\rightarrow 2c_2 + \sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} n c_n x^n - \left(c_0 + \sum_{n=1}^{\infty} c_n x^n\right) = 0$

$\rightarrow 2c_2 - c_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - n c_n - c_n\right]x^n = 0$

$\rightarrow 2c_2 - c_0 = 0 \rightarrow c_2 = \frac{c_0}{2} \rightarrow c_2 = \frac{1}{2}$

$\rightarrow 3\cdot 2 c_3 - 2c_1 = 0 \rightarrow c_3 = \frac{c_1}{3} = 1$

$\rightarrow 4 \cdot 3 c_4 - 3 c_2 = 0 \rightarrow c_4 = \frac{c_2}{4} = \frac{1}{8}$

$\rightarrow 5 \cdot 4 c_5 - 4 c_3 = 0 \rightarrow c_5 = \frac{c_3}{5} = \frac{1}{5}$

$\Rightarrow$ 5th Maclaurin polynomial for the solution is $1 + 3x + \frac{1}{2}x^2 + x^3 + \frac{1}{8}x^4 + \frac{1}{5}x^5$

Ex 9 Find the first five nonzero terms in the series solution to $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - xy = 0$ with $\frac{dy}{dx}\Big|_{x=0} = 1$, $y(0) = 1$.